电子科技大学
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Game-Theoretic Modeling of Network Formation

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2017.06.28

Outline

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- ✓ Stability
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  4. The existence of stability
- ✓ Stochastic Strategic Models
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  2. Farsightedness
- ✓ Directed Networks
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01 Two simple Games

Two simple Games

Why use game theory? There are many settings where links are formed in a cognizant manner, especially in applications where the individual nodes are firms, organizations, or countries, which have explicit objectives when it comes to their relationships. Here, game theory provides a way to interpret the formation of particular networks ***from personal angle****.*

Two simple Games

An Extensive Form Game [Aumann and Myerson]:

- *First fix an order over all possible links, denoted* $(i_1 j_1, \ldots, i_K j_K)$;
- *When the link* $i_k j_k$ *appears in the ordering, the pair of players* i_k *and* j_k *decide on whether or not to form that link, knowing the decisions of all pairs coming before them and forecasting the play that will follow them;*
- *The game continues to move through the remaining unformed links in order, until either all links are formed or there is a round such that all of the links that have not yet formed have been considered and no new links have formed.*

Advantage: always having a pure strategy subgame perfect equilibrium.

Disadvantage: the game can be very difficult to solve, even in very simple settings with only a few players.

Two simple Games

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A Simultaneous Link-Announcement Game [Myerson]:

Each player simultaneously announces the set of players with whom he or she wishes to be linked. The links that are formed are those such that both of the players involved in the link named each other.

More formally,

- *The strategy space of player i is* $S_i = 2^{N\setminus\{i\}}$;
- *The profile of strategies is* $s \in S_1 \times S_2 \times \cdots \times S_n$;
- *The network that forms is* $g(s) = \{ij \mid i \in S_j \text{ and } j \in S_i\}$.

This game is much easier to describe than the Aumann and Myerson extensive form, and it avoids inducing a priori asymmetries between the players or links. Thus, we will introduce the ***Stability*** *concept by using this game, where something is really interesting!*

02 Stability

Stability

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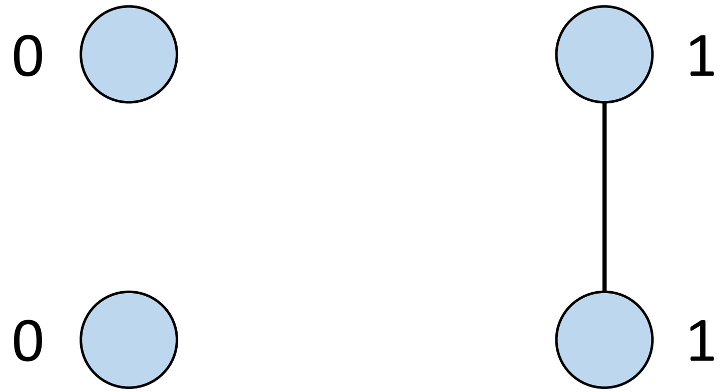
Which one is nash equilibrium?



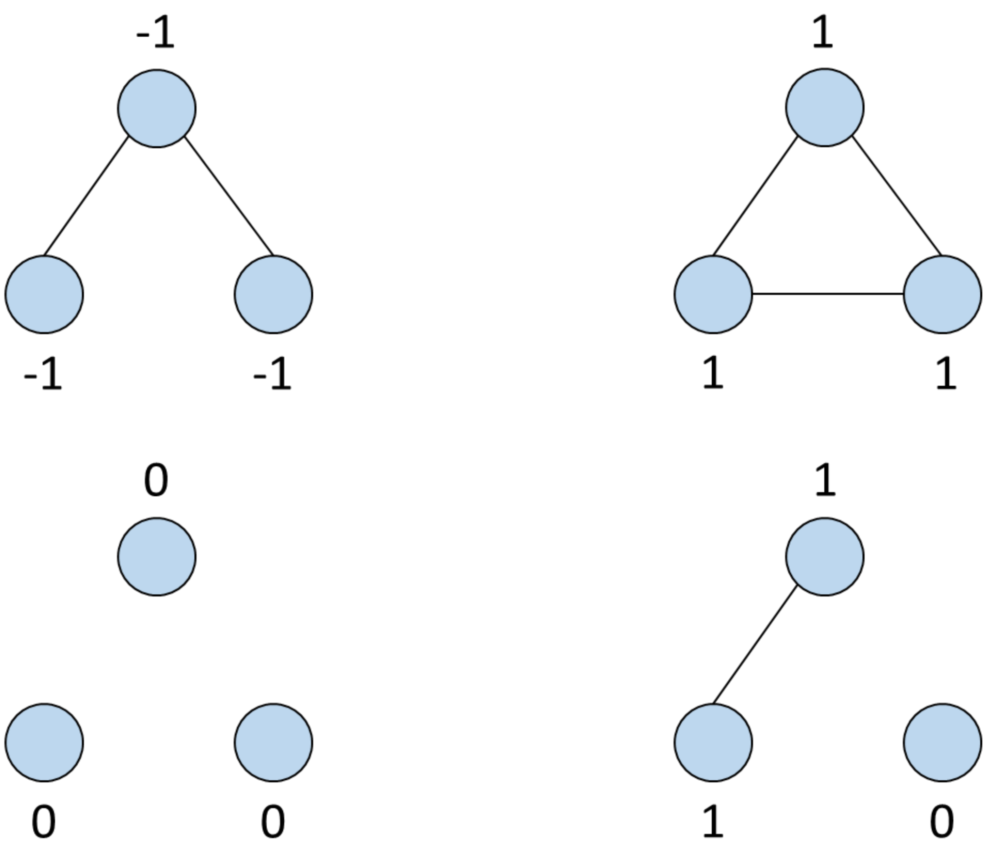
Two networks of the Link Announcement Game

Both networks are Nash Equilibria!

*Nash Equilibrium: if each player has chosen a strategy and **no** player can benefit by changing strategies **while the other players keep theirs unchanged**, then the current set of strategy choices and the corresponding payoffs constitutes a Nash equilibrium.*

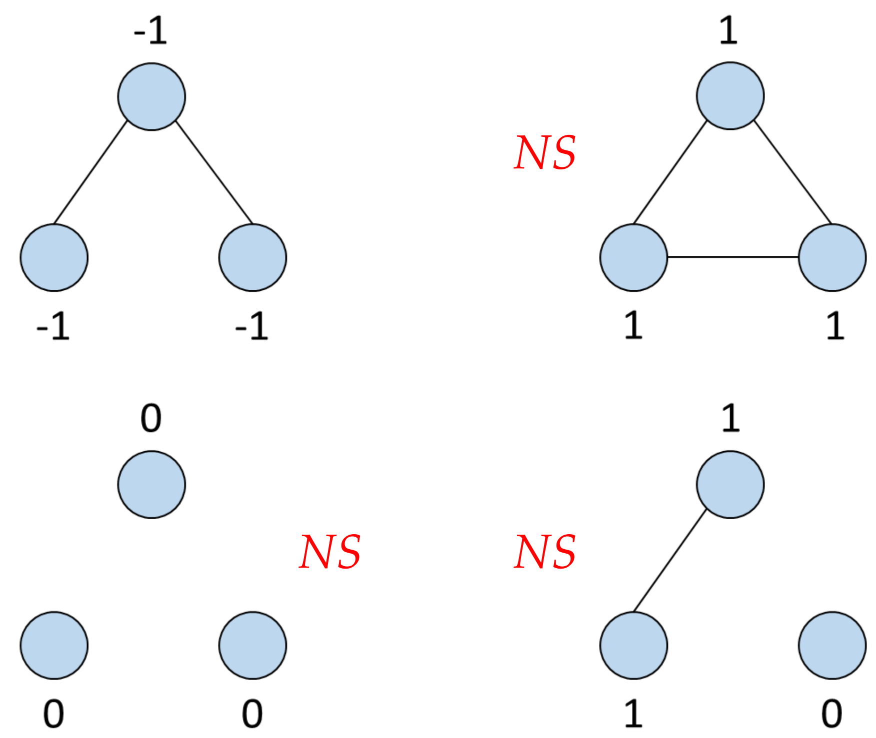
Another example:

Which ones are nash equilibria?



Another example:

*The main drawback of the game that it has **too many Nash equilibria**, including some which are easily seen to be unreasonable, such as the empty network which is always a Nash equilibrium, regardless of the payoffs.*



Thus, new refined solutions need to be proposed!

Stability

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The reason for the inadequacy of Nash equilibrium:

It allows players to refuse to form links, and thus effectively to "delete" links, but it does not capture the fact that it may be mutually advantageous for two players to form a new relationship.

在一般的无向图模型中，增边需要双方达成一致，而减边则仅需一个人的意愿。

Thus some coalitional considerations have to be considered. If we consider one pair of players:

Pairwise Stability

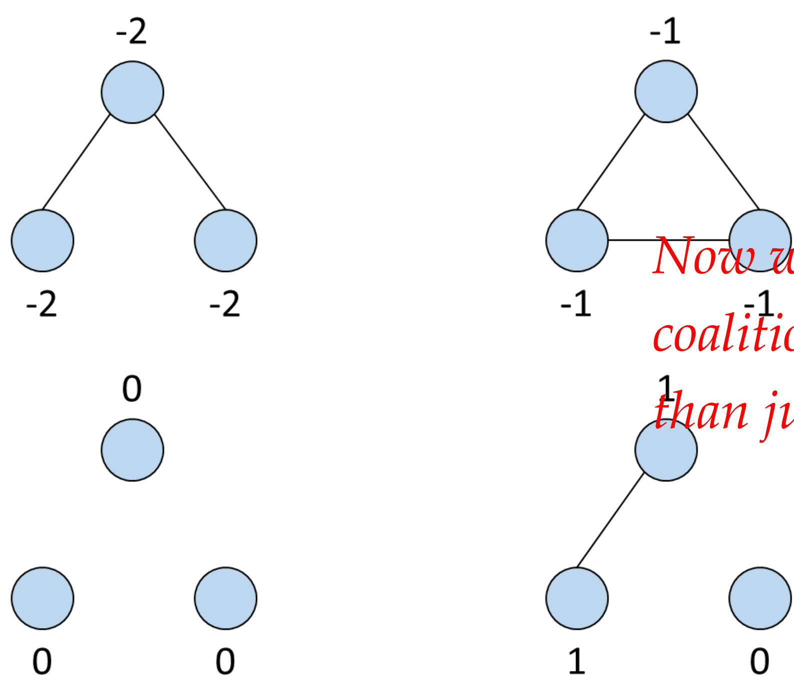
It restricts attention to changes of one link at a time.

Pairwise Stability

A network g is pairwise stable if

(i) *for all* $ij \in g$, $u_i(g) \geq u_i(g - ij)$ *and* $u_j(g) \geq u_j(g - ij)$, *and*

(ii) *for all* $ij \notin g$, $u_i(g + ij) > u_i(g)$ *then* $u_j(g + ij) < u_j(g)$.



PS

Now we can consider larger coalitions of players other than just two people!

PS

Stability

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A network $g' \in G$ *is obtainable from* $g \in G$ *via deviations by* $S \subset N$ *if:*

(i) $ij \in g'$ *and* $ij \notin g$ *implies* $\{i, j\} \subset S$ *, and*

(ii) $ij \in g$ *and* $ij \notin g'$ *implies* $\{i, j\} \cap S \neq \phi$.

即g'能由g通过S的背离得到，只要满足

（a）在g'不在g的边的两端必须都包含在S中，表由g到g'的增边；

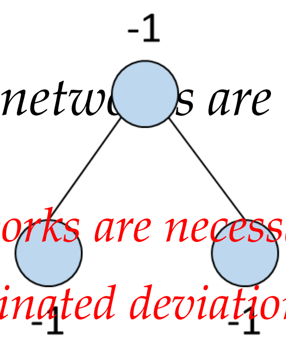
（b）而在g不在g'中的边至少有一端在S中，表由g到g'的减边。

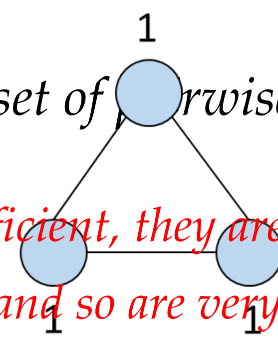
The above definition identifies changes in a network that can be made by a coalition S ***without the consent of any players outside of S***.

Strong Stability

A network g is strongly stable with respect to a profile of utility functions $u = (u_1, \ldots, u_n)$, *if for* ***any*** $S \subset N$, g' *that is obtainable from* g *via deviations by* S, *and* $i \in S$ *such that* $u_i(g') > u_i(g)$, *there exists* $j \in S$ *such that* $u_j(g') < u_j(g)$.

The strongly stable networks are always a subset of pairwise stable networks.

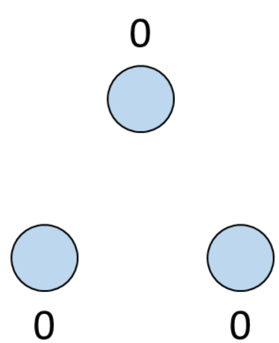




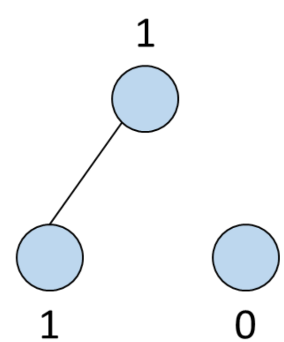
NS, PS, SS

Strongly stable networks are necessarily Pareto efficient, they are immune to all sorts of coordinated deviations by players, and so are very robust.

Only the complete network is strongly stable!



NS



NS, PS

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The Existence of Stable Networks:

While the the link-announcement game always has an equilibrium, this is due to the fact ***that there is always a trivial equilibrium where no links form*** *because no player expects any other player to be willing to form a link. Once we move to refinements for which the empty network is not always stable, such as Pairwise Stability, Strong Stability, or other such refinements,* ***existence is not always guaranteed****.*

The Existence of Stable Networks:

A network g' defeats an adjacent network g if either:

(i) $g' = g - ij$ *and* $u_i(g') > u_i(g)$, *or if*

(ii) $g' = g + ij$ *and* $u_i(g') \geq u_i(g)$ *and* $u_j(g') \geq u_j(g)$, *with at least one inequality holding strictly.*

Two networks are adjacent if they differ by only one link.

Thus a network is pairwise stable if and only if it is not defeated by an (adjacent) network.

Improving path:

a sequence of distinct networks $\{g_1, g_2, \dots, g_K\}$, *such that each network* g_k *with* $k < K$ *is adjacent to and defeated by the subsequent network* g_{K+1}.

The Existence of Stable Networks:

Proposition1:

a network is pairwise stable if and only if it has no improving paths emanating from it.

*Note: The notion of improving paths is a **myopic** one, in that the agents involved in adding or deleting links are doing so without forecasting how their actions might affect the evolution of the process.*

Proposition2:

If there does not exist any pairwise stable network, then there must exist at least one improving cycle. (Conversely, the existence of pairwise stability comes from ruling out improving cycles)

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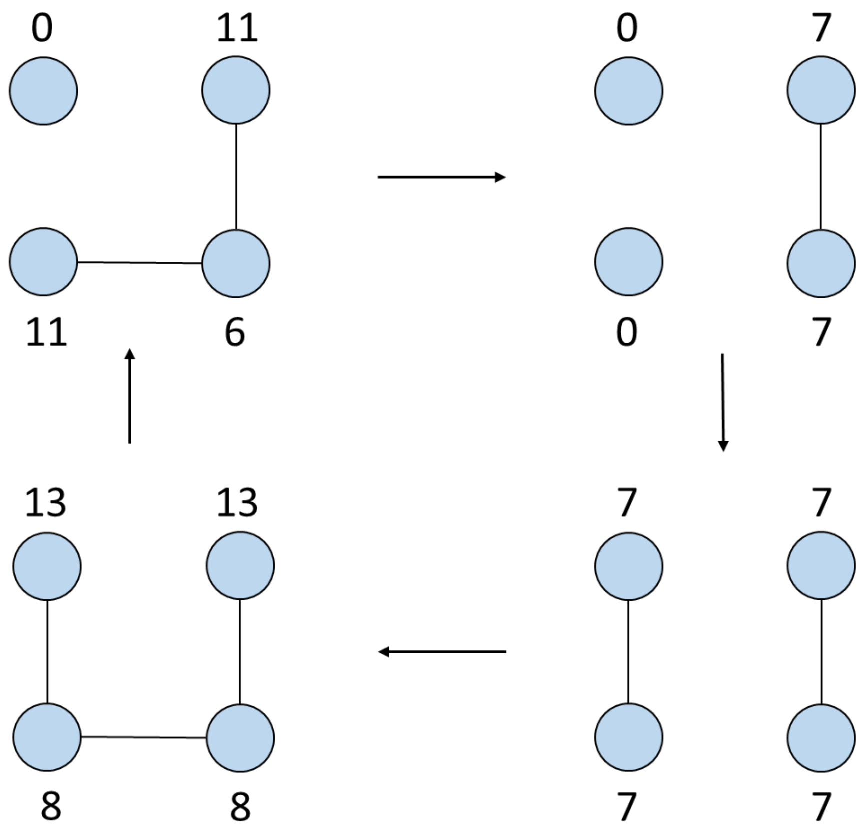
The Existence of Stable Networks:

One example for Proposition2:



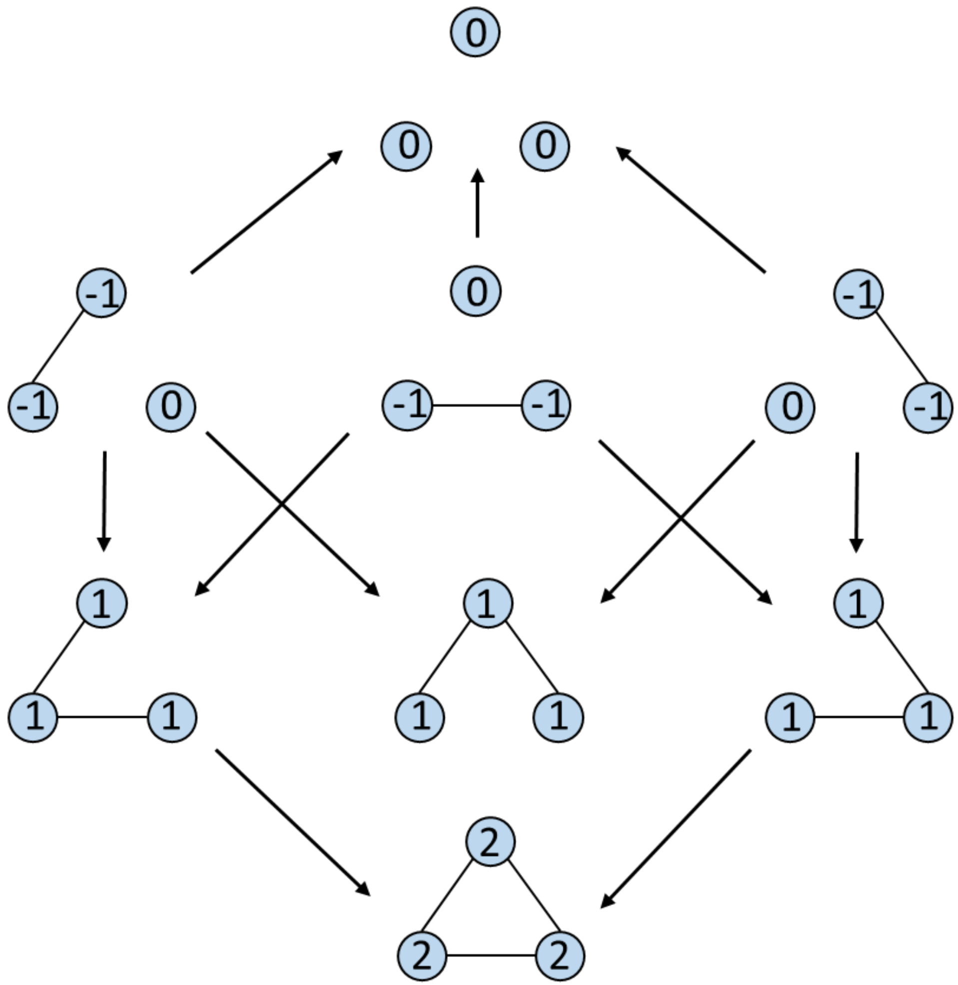
Payoff function:

- *The benefit for zero, one, two, three links are 0, 12, 16, 18.*
- *The cost for one link is 5.*

03 *Stochastic Strategic Models*

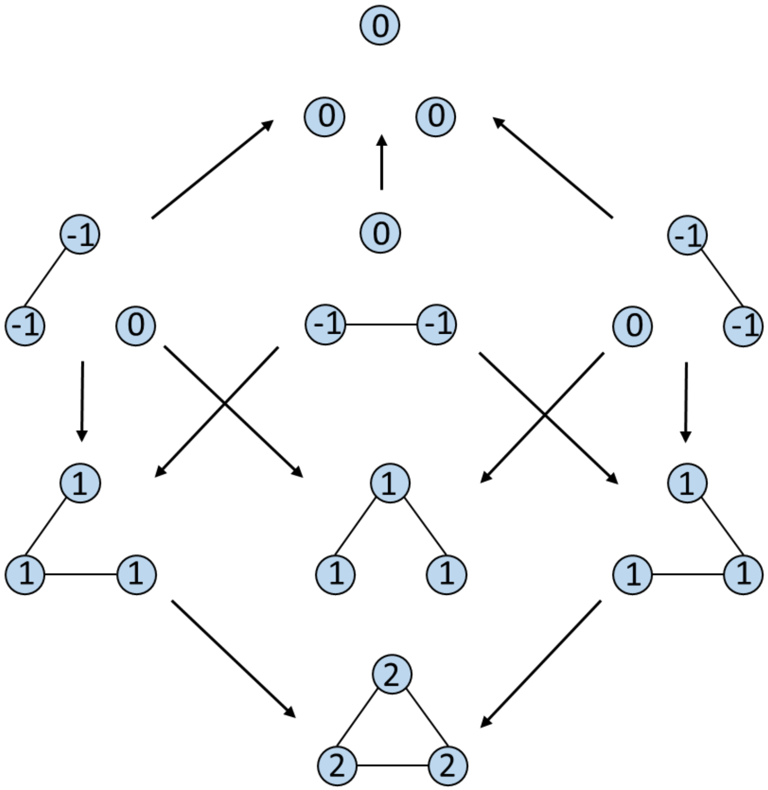
Stochastic Strategic Models

An Example



*Two Pairwise Stable Networks, where Improving Paths can get **stuck** at the Empty Network.*

Stochastic Strategic Models



There are two different ways in which this process might get "unstuck":

- *Trembles (wrong behaviors);*
- *Farsightedness.*

One simply introduces some randomness into the process, while the other relies on rational and forward-looking players.

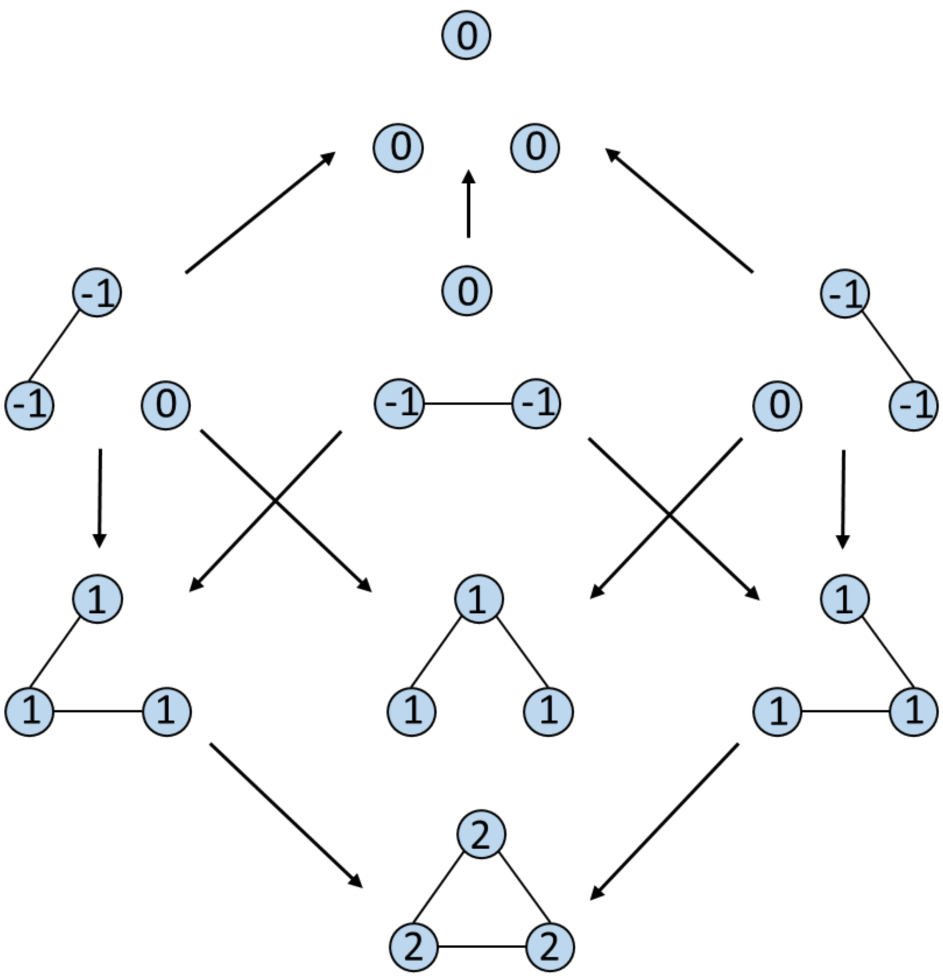
Trembles：

The improving path process [Jackson and Watts]:

- *Start at any network $g \in G$.*
- *At each time, $t \in \{1,2, \dots\}$ a link* ***ij*** *is randomly identified, with each link having an equal probability of being identified. Whether to add or delete this link* ***ij*** *is also dependent on these two persons' utilities.*
- *With a probability of $1 - \varepsilon$ the intent of the players (to add a link, to delete a link, or to leave the network as it is) is carried out, and with probability $\varepsilon > 0$ the reverse occurs.*

In fact, this process is now a finite state, aperiodic, irreducible Markov chain, thus it has ***a steady-state distribution****.*

Trembles:



Its transition matrix:

$$\Pi(\varepsilon) = \begin{pmatrix} 1-\varepsilon & \varepsilon & 0 & 0 \\ \frac{1-\varepsilon}{3} & \varepsilon & \frac{2(1-\varepsilon)}{3} & 0 \\ 0 & \frac{2\varepsilon}{3} & \frac{2-\varepsilon}{3} & \frac{1-\varepsilon}{3} \\ 0 & 0 & \varepsilon & 1-\varepsilon \end{pmatrix}$$

Thus we do converge to the most reasonable network !

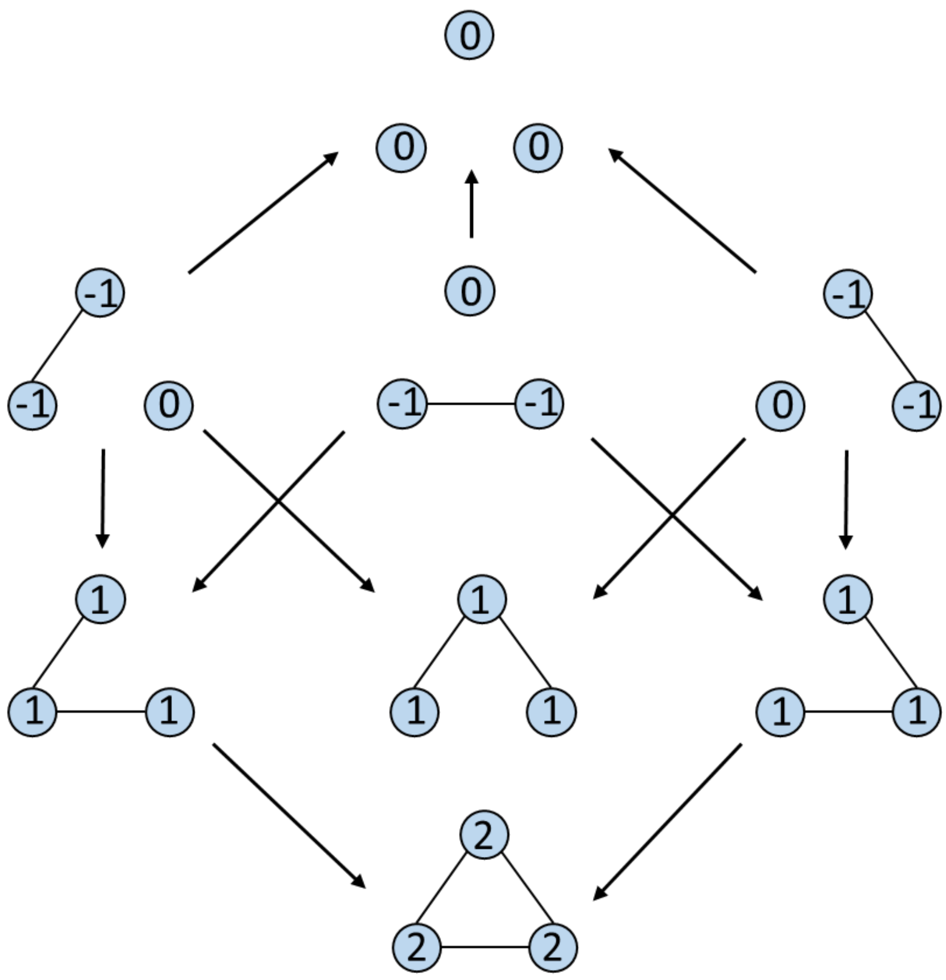
The steady-state distribution:

$$\mu(\varepsilon) = \left(\frac{\varepsilon(1-\varepsilon)}{1+2\varepsilon}, \frac{3\varepsilon^2}{1+2\varepsilon}, \frac{3\varepsilon(1-\varepsilon)}{1+2\varepsilon}, \frac{(1-\varepsilon)^2}{1+2\varepsilon} \right)$$

As we let ε go to 0, $\mu(\varepsilon)$ tends to

(0, 0, 0 ,1)

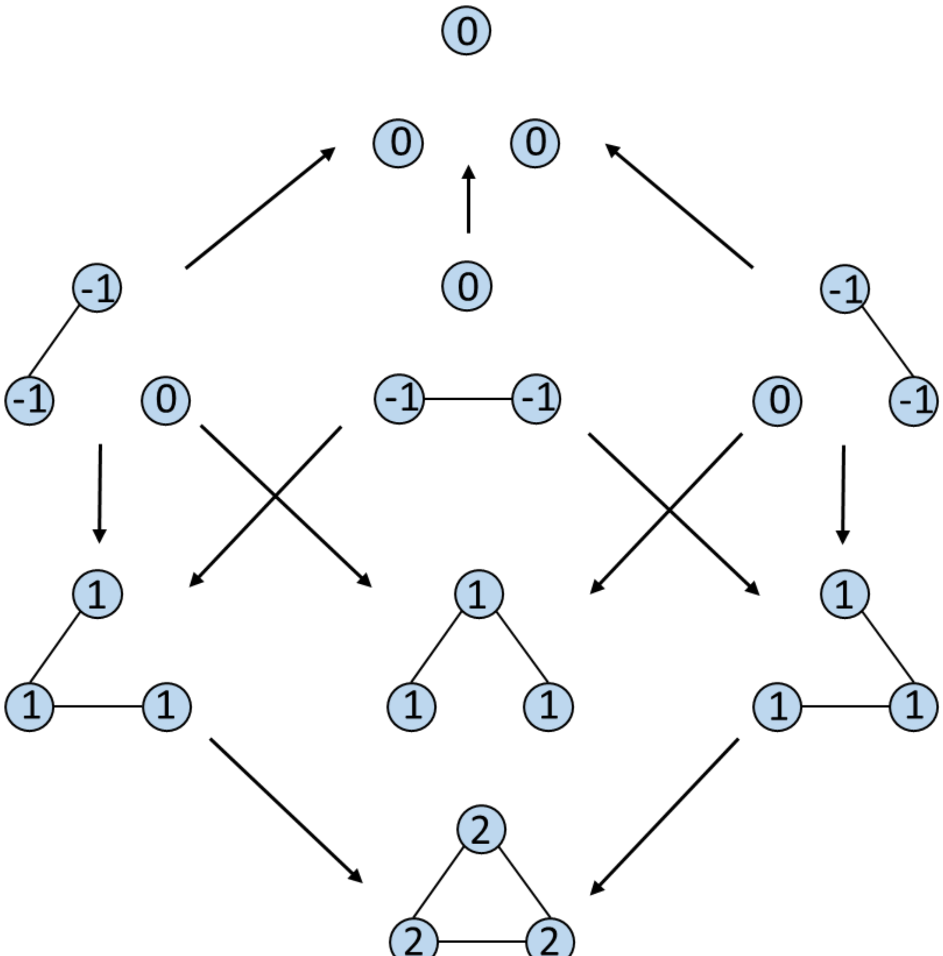
Trembles：



So why it happens?

It needs a least two errors for the complete network going to the empty network, but the negative direction only needs one error.

Trembles:



By contrast, let us remove any error, the transition matrix is:

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \varepsilon & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Its steady-state distributions are:

$(\boldsymbol{a}, \mathbf{0}, \mathbf{0}, \mathbf{1} - \boldsymbol{a})$ *for any* $\boldsymbol{a} \in [\mathbf{0}, \mathbf{1}]$

The process without any mutations does not discriminate between the empty and complete networks.

Farsightedness：

In the definition of improving path changes from one network to the next are improving for the players involved, but without anticipating the subsequent changes that will occur along the path. In contrast, the idea of a farsighted improving path captures the notion that the players anticipate the further changes along the path and compare the ending network and the current one.

Farsightedness：

A network $\boldsymbol{g'}$ ***is improving for S relative to*** $\boldsymbol{g}$ *if it is weakly preferred by all players in S to g, with strict preference holding for at least one player in S*

A ***farsighted improving path****:*

consider a sequence of networks $g_1, g_2, \ldots, g_K$*, and a corresponding sequence* $S_1, S_2, \ldots, S_{K-1}$*, such that* g_{K+1} *is obtainable from* g_K *via deviations by* S_K*. A sequence is a farsighted improving path if, for each k, the ending network* g_{K+1} *improving for* S_K *relative to* g_K*.*

Farsightedness:

A network g is ***farsightedly strongly stable*** *if there is no farsighted improving path from g to some other network g'.*

However, it's such a demanding requirement that there's probably no farsightedly strongly stable networks in many games.

Thus, we use ***consistent set*** *to describe stability:*

A set $A \subset G$ is consistent if for each $g \in A$, and g' obtainable from g via deviations by some $S \subset N$, either $g' \in A$ and g' is not improving for S, or there exists a farsighted improving path from g' to some $g'' \in A$ such that g'' is not improving for S.

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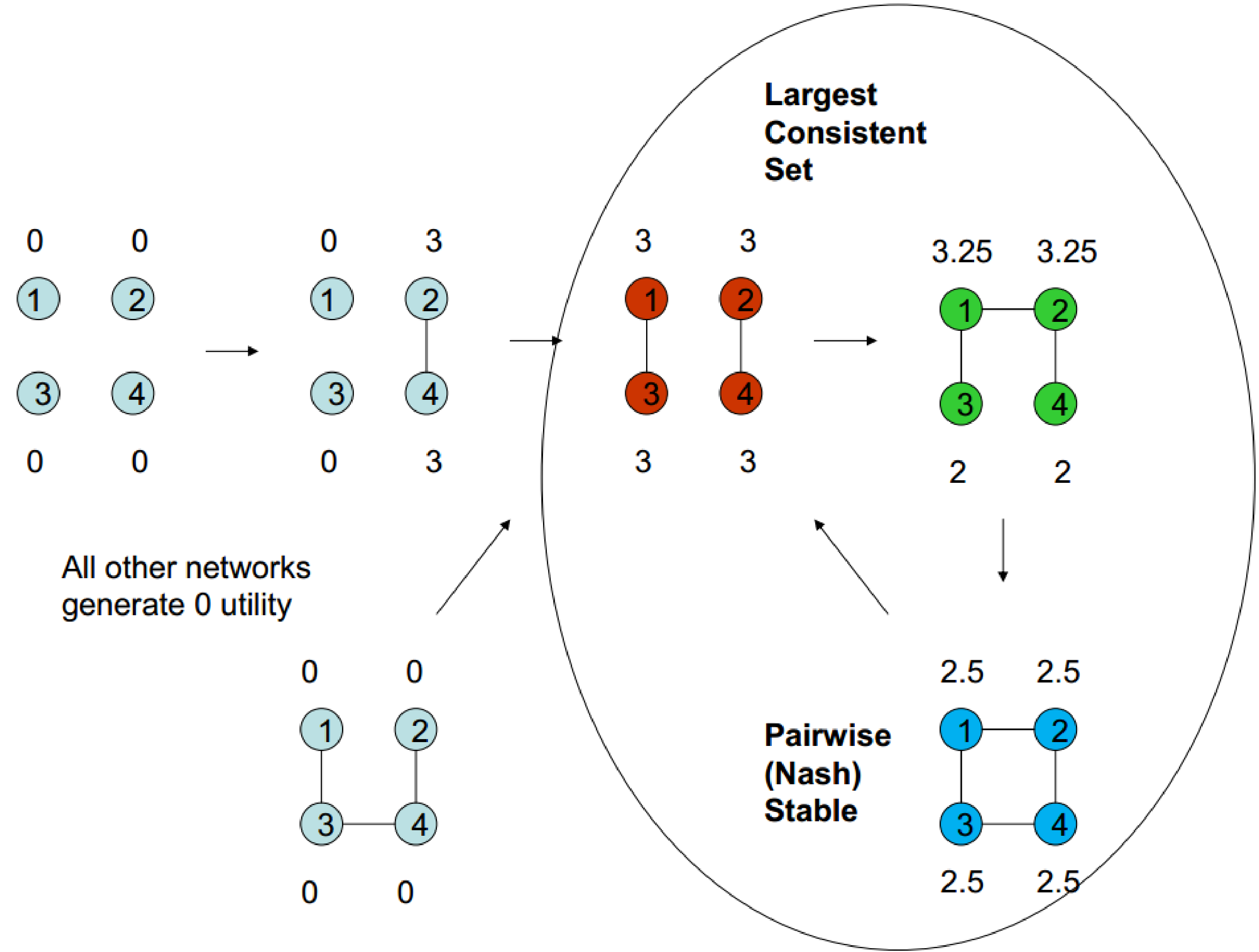
Farsightedness:

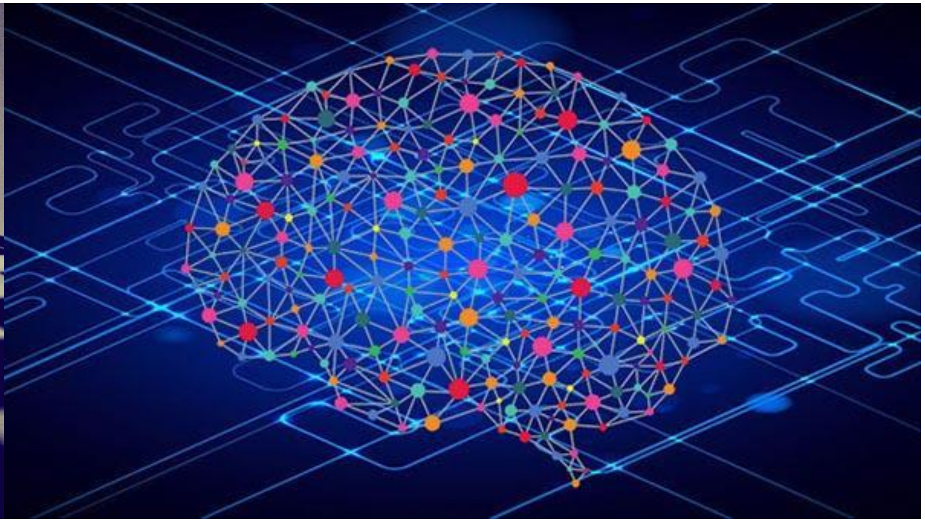
An example:



Farsightedness:

Proposition3:

Consider any N and profile of preferences. There exists a unique largest consistent set (so that every consistent set is a subset of it and it is consistent), and this set is nonempty.



04 Directed Networks

Directed Networks

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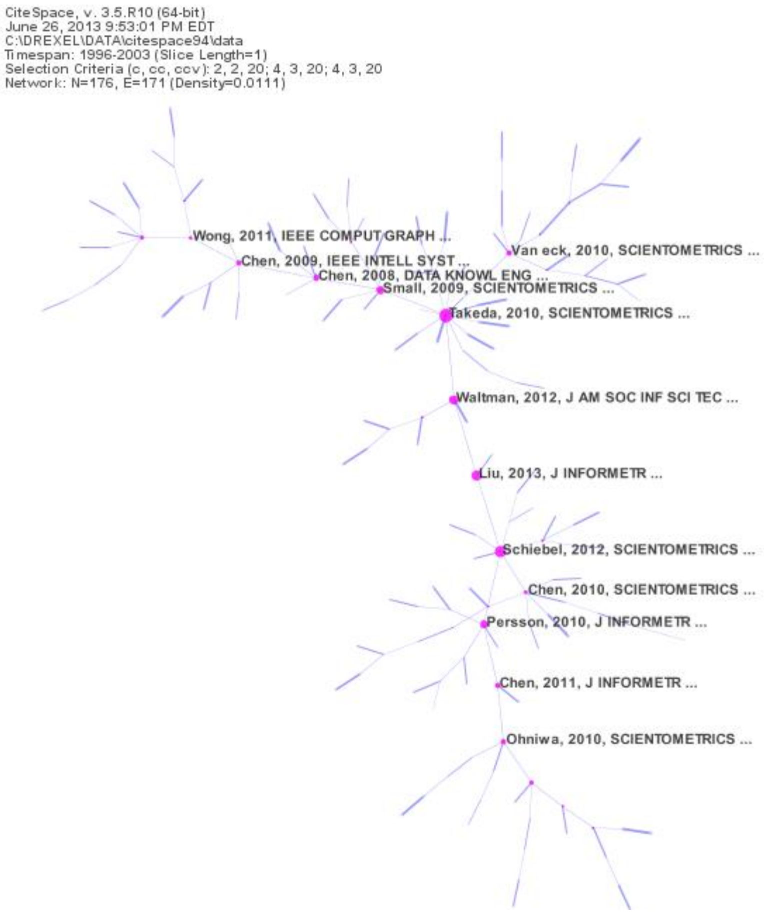
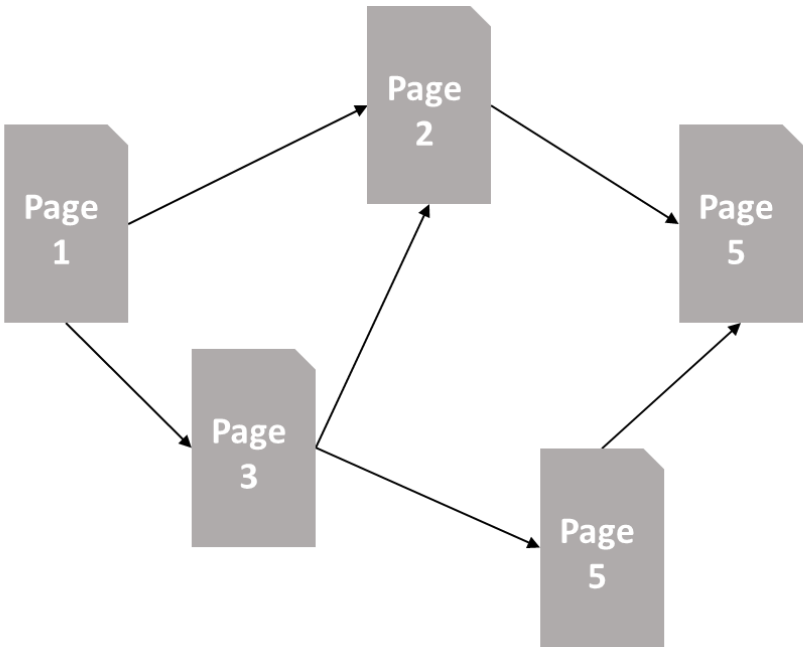
Some applications where links can be formed unilaterally:





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Directed Networks

How model the directed network formation?

> *An simple way is to have each player list the set of directed links that he or she wishes to form (and the player can only list links from him or herself to another player), and then have the resulting network be the union of the listed links.*

A network g' is obtainable from a network g by player i if $g'_{kj} \neq g_{kj}$ implies that $k = i$. (两个网络的不同完全来自于局中人i的连接策略的不同)

A directed network g is ***directed Nash stable*** *if $u_i(g) \geq u_i(g')$ for each i and all networks g' that are obtainable from network g by player i.*

Directed Networks

The utility function:

1. Two-way flow:

$$u_i(g) = \sum_{j \neq i : j \in N^{n-1}(\hat{g})} b\big(l_{ij}(\hat{g})\big) - d_i(g)c$$

$\hat{g}$: *the undirected network obtained by allowing an (undirected) link to be present whenever there is a directed link present in* g.

b : the net benefit from direct and indirect connections, such as b(k)=δ^k.

$l_{ij}(\cdot)$: *the shortest path length between i and j.*

$d_i(g)$: *the degree of i.*

c: the fixed cost.

Directed Networks

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The utility function:

1. Two-way flow:

$$u_i(g) = \sum_{j \neq i : j \in N^{n-1}(\hat{g})} b\big(l_{ij}(\hat{g})\big) - d_i(g)c$$

Both sides benefit from the link being present, but only one side pay the cost.

Directed Networks

The utility function:

1. *Two-way flow:* $$u_i(g) = \sum_{j \neq i: j \in N^{n-1}(\hat{g})} b\big(l_{ij}(\hat{g})\big) - d_i(g)c$$

Proposition4(Stability):

(i) *If $c<b(1)-b(2)$, then the directed Nash stable networks are those that have one directed link between each pair of players;*

(ii) *If $b(1)-b(2)<c<b(1)$, then any directed star encompassing all nodes is directed Nash stable and for some parameters there are other directed Nash stable networks.*

(iii) *If $b(1)<c<b(1)+\frac{n-2}{2}b(2)$, then peripherally sponsored stars*[1] *are Nash stable and so are other networks (e.g., the empty network),*

(iv) *If $b(1)+\frac{n-2}{2}b(2)<c$, then only the empty network is directed Nash stable.*

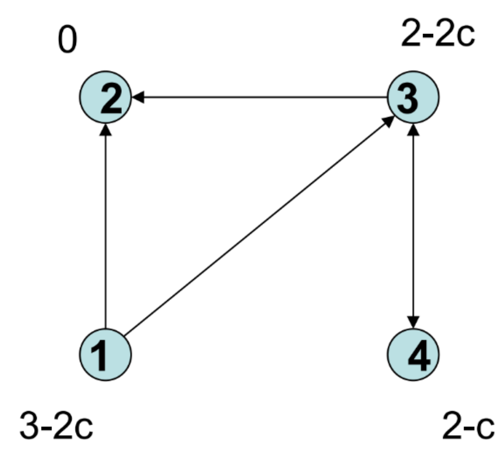
The utility function:

2. One-way flow:

In one particular situation, let an arbitrarily distant connection provides the same benefit as a direct connection:

$$u_i(g) = R_i(g) - d_i(g)c$$

$R_i(g)$ *: the number of other players who can be reached from i via a directed-path in g.*



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The utility function:

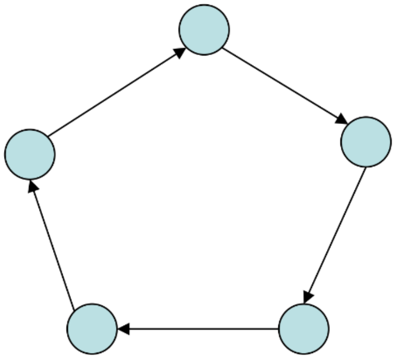
2. *One-way flow:* $u_i(g) = R_i(g) - d_i(g)c$

Proposition5(Stability):

(i) If c<1, then n-player wheels[1] are the (only) strictly Nash stable networks;

(ii) If 1<c<n-1, then n-player wheels and empty networks are the (only) strictly Nash stable networks;

(iii) If c>n-1, then the empty network is the unique strictly Nash stable network.



A Wheel on Five Nodes

05 Conclusion

Two simple games: Link-Announcement Game

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Too many Nash equilibria, thus refined solutions

↓

Two players(one pair) —— Pairwise stability;
Larger coalition of players —— Strong stability.

↓

The Existence of Stable Networks: Improving path(myopic)

↓

Stochastic Strategic Models: Trembles and Farsightedness

↓

Directed networks: two-way flow and one-way flow (directed nash stable)

Thanks!